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A dynamical objection to the inversion of the Earth on its spin axis

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Abstract. Warlow has proposed a sudden inversion of the Earth on its spin axis to explain geomagnetic reversals and other phenomena. A complete analysis of his dynamical equations shows that the torque required for this inversion is 200 times greater than the value he derives, and cannot be produced by the gravitational attraction of any known cosmic body in the solar system making a close passage past Earth. Specifically, Warlow's inversion requires a 417 Earth mass body passing at two Earth radii.

1. Introduction

Warlow (1978) has proposed that a relatively small torque applied to the Earth can cause it to invert on its spin axis in tippe-top fashion. The direction of spin in inertial space would not be changed much by the application of the torque, but the Earth's geographic north and south poles would exchange places on the inertial spin axis. Relative to the Earth's surface, the direction of spin would have changed. Because of the interest in Warlow's hypothesis among proponents of a catastrophic evolution of the Earth, a detailed examination of the inversion dynamics has been performed. Three dynamical assumptions underlie his hypothesis, as follows.

(i) The Earth may be treated as a perfectly rigid body.

(ii) The laws of Newtonian mechanics may be applied to the problem; specifically, Euler's dynamical (differential) equations hold, i.e. Warlow's equations (5)–(7).

(iii) The gravitational torque applied to the Earth by a passing cosmic body is sufficient to cause the desired inversion of the Earth on its spin axis.

This note presents a rigorous calculation of the torques and shows the specific places where Warlow has neglected major contributions to the torques required by Euler's dynamical equations.

2. Basic theory

For the exposition, Warlow (1978, § 7) adopts a set of Cartesian axes $x_1\, x_2\, x_3$ fixed in the solid Earth with origin 0 at the Earth's mass centre. These axes as drawn in Warlow's figure 4 form a left-handed coordinate system. The $+x_1$ axis points northward along the principal moment-of-inertia axis of the Earth (axis of figure) that is near the normal (regular) spin axis. x_2 and x_3 are equatorial axes.

Warlow assumes that the Earth spins about an inertially fixed axis $0Z$ at a constant angular rate $\dot{\phi}$ given by

$$\dot{\phi} = 2\pi \text{ rad/day}. \tag{1}$$

The $0Z$ axis lies in the x_1x_2 plane and makes an angle θ with the x_1 axis and an angle $(\frac{1}{2}\pi - \theta)$ with the x_2 axis. During the inversion, θ changes from 0 to π radians due to an additional, simultaneous rotation of the Earth about the x_3 axis at the angular rate $\dot{\theta}$. Warlow calls this additional rotation 'fast precession'.

The resulting angular rates about body-fixed axes are

$$\Omega_1 = \dot{\phi} \cos \theta \tag{2}$$

$$\Omega_2 = \dot{\phi} \sin \theta \tag{3}$$

$$\Omega_3 = \dot{\theta}. \tag{4}$$

Because $\dot{\phi}$ is constant, the time derivatives of equations (2) and (3) give the angular accelerations

$$\dot{\Omega}_1 = -\dot{\phi}\dot{\theta} \sin \theta \tag{5}$$

$$\dot{\Omega}_2 = \dot{\phi}\dot{\theta} \cos \theta. \tag{6}$$

Warlow (1978, § 4, last three paragraphs; § 6 the complete paragraph spanning pp 2120–1) indicates that the entire inversion takes place in about one day. For generality in the equations, we take the time interval for the inversion to be n days. As a consequence, $\dot{\theta}$ must attain a value during that interval of (π/n) rad/day or greater. Because $\dot{\theta}$ is zero both before and after the inversion, an angular acceleration $\dot{\Omega}_3$ must occur about the x_3 axis during the inversion. If $\dot{\Omega}_3$ has a constant positive value during the first half of the inversion and a constant negative value during the second half, $|\dot{\Omega}_3| \sim (4\pi/n^2)$ rad/day^2. $\dot{\theta}$ then has a maximum magnitude of $(2\pi/n)$ rad/day. If $|\dot{\Omega}_3|$ is not constant, even greater values of its magnitude will occur. The use of these estimates and equations (1), (5) and (6) gives the estimates

$$\begin{aligned} \dot{\Omega}_1 &\sim -(2\pi^2/n) \sin \theta \text{ rad/day}^2 \\ \dot{\Omega}_2 &\sim (2\pi^2/n) \cos \theta \text{ rad/day}^2 \\ |\dot{\Omega}_3| &\geq (4\pi/n^2) \text{ rad/day}^2 \end{aligned} \tag{7}$$

for the angular accelerations about body axes during the inversion. Note that different acceleration components depend on different powers of n.

Let I_1, I_2, I_3 be the moments of inertia of the Earth about the axes x_1, x_2, x_3, respectively. We may assume $I_2 = I_3$ for simplicity in Euler's dynamical equations (Warlow's equations (5)–(7)). For the Earth, $(I_1 - I_2) \simeq (I_1 - I_3) \sim I_1/300$ (Jeffreys 1970, § 4.03).

3. Oversights in Warlow's theory

From Euler's dynamical equations, Warlow obtains his equation (8)

$$I_3\dot{\Omega}_3 = \tau_3 - \tfrac{1}{2}(I_1 - I_2)\dot{\phi}^2 \sin 2\theta \tag{8}$$

where τ_3 is the external torque along the x_3 axis. We have no disagreement with this equation or its derivation.

But Warlow errs when he states in the sentence following his equation (8) (Warlow 1978, p 2123) that 'in order to initiate and just maintain fast-precession rotation, we require τ_3 to just exceed the value of the second term on the right-hand side of (8) . . .', that is, he only requires

$$\tau_3 \geqslant \tfrac{1}{2}(I_1 - I_2)\dot{\phi}^2 \sin 2\theta \qquad \text{(incorrect)}. \tag{9}$$

This relation says that

$$\tau_3 \sim (I_1/300)2\pi^2 \text{ rad/day}^2 \qquad \text{(incorrect)}. \tag{10}$$

But Warlow has overlooked the importance of the angular acceleration $\dot{\Omega}_3$ on the left side of equation (8). Using its magnitude from equation (7) gives the result

$$\tau_3 \sim I_3(4\pi/n^2) \text{ rad/day}^2 \tag{11}$$

a quantity 200 times larger than Warlow's lower limit when the inversion takes place in one day ($n = 1$). Because of this oversight, Warlow has *significantly underestimated* the torque required to produce an inversion.

A second error occurs when he assumes in the sentence following his equation (7) (Warlow 1978, p 2123) that 'For precession about $0x_3$ only, *we may neglect* τ_1 *and* τ_2 and consider τ_3 alone . . .' (emphasis added). Since $I_2 = I_3$, Euler's dynamical equation for the x_1 axis (Warlow's equation (5)) and our equation (7) give

$$\tau_1 = I_1\dot{\Omega}_1 = -I_1(2\pi^2/n) \sin \theta \text{ rad/day}^2 \tag{12}$$

which shows that τ_1 is of the same order as τ_3 and cannot be neglected. (If n is much greater than 1, $|\tau_1| > |\tau_3|$.) This is Warlow's most important error because a gravitational torque about the Earth's x_1 axis (axis of figure) is very difficult to produce, as the next section shows. A similar analysis for the x_2 axis gives

$$\tau_2 = I_2\dot{\Omega}_2 = I_2(2\pi^2/n) \cos \theta \text{ rad/day}^2 \tag{13}$$

which is also of the same order as τ_3.

This second error arises because Warlow has explicitly assumed that the precession axis has the same direction as the applied torque, an assumption that we have just shown is incorrect. A familiar textbook example where this assumption does not hold is gyroscopic precession.

A third error arises from Warlow's appeal to 'precessional momentum' to carry the inversion through to completion if the torque applied to the Earth falls below the required magnitude (Warlow 1978, p 2123 near the end). There is no such thing as 'precessional momentum'; such a concept is inconsistent with Euler's dynamical equations, Warlow's equations (5)–(7). Once the inversion scheme has been specified, these equations fully specify the time dependence of the three torque components that must be applied to the Earth to cause the inversion. If the torque goes to zero before the inversion is completed, these equations require that the 'fast precession' cease and that the Earth resume free-body nutation in which θ has a large, constant value. 'Fast precession' can occur in the absence of a specified torque component only if Newton's laws are violated, and this would be contrary to assumption (ii) of Warlow's hypothesis.

4. Gravitational torques

We now determine if a close encounter by a cosmic body with the Earth can give a gravitational torque sufficient to invert the Earth on its spin axis. We first write the potential energy V of a cosmic body of mass m_c in the Earth's gravitational field. For a body at a centre-to-centre distance r from the Earth's centre, at a geocentric latitude b and geocentric longitude λ measured east from Greenwich,

$$V = -\frac{Gm_E m_c}{r}[1 - J_2(a_E/r)^2(\tfrac{3}{2}\sin^2 b - \tfrac{1}{2}) + J_{2,2}(a_E/r)^2 3\cos^2 b\cos 2(\lambda - \lambda_{2,2})] \quad (14)$$

to order r^{-3}. Here $a_E = 6378$ km, the Earth's equatorial radius; $G = 6.67\times10^{-11}$ Nm^2/kg^2, the universal constant of gravitation; $m_E = 5.97\times10^{24}$ kg, the Earth's mass; $J_2 = 1.082\times10^{-3}$, the coefficient of the second zonal gravitational harmonic; $J_{2,2} = 1.82\times10^{-6}$, the coefficient of the second sectoral gravitational harmonic (Wagner 1966) and $\lambda_{2,2}$ = phase angle for that harmonic, approximately $-15°$. The gravitational torque on the Earth produced by the body has a component along the x_1 axis given by

$$\begin{aligned}\tau_1 &= (\partial V/\partial\lambda)\\ &= 6\frac{Gm_E m_c}{r^3}J_{2,2}a_E^2\cos^2 b\sin 2(\lambda - \lambda_{2,2}).\end{aligned} \quad (15)$$

The torque components along the x_2 and x_3 axes are proportional to $\partial V/\partial b$, so the J_2 term in equation (14) contributes to these components and they can easily exceed τ_1 in magnitude by a factor of about 100. Since we found in the previous section that τ_1, τ_2 and τ_3 must be of the same order of magnitude to produce an inversion, we focus our attention on whether the generally smaller component τ_1 can ever have the required magnitude even momentarily. This component is largest when the cosmic body is in the Earth's equatorial plane ($b = 0$) at a longitude that makes the sine term equal 1. Setting this largest value equal to the τ_1 magnitude found previously from equation (12) gives

$$I_1\frac{2\pi^2}{n}\frac{\text{rad}}{(86\,400\text{ s})^2} = 6\frac{Gm_E m_c}{r^3}J_{2,2}a_E^2. \quad (16)$$

Jeffreys (1970, § 4.04) gives $I_1 = 0.331\, m_E a_E^2$ so the numerical evaluation of equation (16) gives

$$m_c/r^3 = 52.2m_E/(a_E^2 n) \quad (17)$$

as a condition on the mass and distance of the cosmic body to produce a torque of the required magnitude.

For an inversion taking place in one day ($n = 1$) caused by a cosmic body with the Earth's radius which just missed hitting the Earth's surface (i.e. $r = 2a_E$), equation (17) gives $m_c = 417m_E$ for the mass of the cosmic body. This is greater than Jupiter's mass ($318m_E$). Cosmic bodies at greater distances would require still greater masses to produce the required gravitational torque. Even if there were such a compact cosmic body that just missed hitting the Earth, its motion past the Earth (with a velocity greater than Earth's escape velocity) would not keep it within $2a_E$ of the Earth's centre for an hour, let alone the 24 h that the torque must act to invert the Earth on its spin axis. This conclusion that gravitational torques cannot cause an inversion contradicts Warlow's assumption (iii). Such a close approach of a massive body to the Earth would also greatly change Earth's orbit about the sun.

An inversion taking a longer time (n days) would require a less massive cosmic body, but one which must stay close to the Earth for a much longer time span, an increased difficulty.

5. Torque-impulse required for any inversion scheme

We now address the question of whether a different precessional scheme could invert the Earth on its spin axis without requiring a large applied torque. For the angular rate about the x_1 axis, any inversion actually requires that $\Omega_1 = +\omega$ at a time t_1 just before the inversion starts and $\Omega_1 = -\omega$ at a time t_2 when the inversion ends, where $\omega = 2\pi$ rad/day. With the assumption that $I_2 \simeq I_3$, Warlow's equation (5) gives

$$\int_{t_1}^{t_2} \tau_1 \, \mathrm{d}t \simeq I_1[(\Omega_1)_{t_2} - (\Omega_1)_{t_1}] = -2I_1\omega \tag{18}$$

for the torque impulse along the x_1 axis. This equation shows that the torque impulse required by any inversion scheme is the same as the torque impulse required for a spin reversal, that is to just stop the Earth's spin about its axis of figure (the x_1 axis) and then restore the spin to the same magnitude but with the opposite sense. Inverting the Earth on its spin axis gravitationally thus offers no theoretical advantages to catastrophic evolution.

6. Conclusion

We have investigated the rotational dynamics of an inversion of the Earth on its spin axis subject to the assumptions of Warlow's hypothesis, and found that no inversion scheme allows a passing planetary body to exert a sufficient gravitational torque about the Earth's axis of figure to cause the inversion. Three separate errors were identified in Warlow's analysis of the problem. If such an inversion ever did occur, its mechanism remains unexplained.

Those who would now appeal to electromagnetic forces from a cosmic body to invert the Earth have the burden of demonstrating *quantitatively* that (i) an electromagnetic force can produce torques of sufficient magnitude, and that (ii) the torque components along the x_1 x_2 x_3 axes have the proper time dependence to produce an inversion.

References

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